Name: ______________________________ Date: ________________________________

PSTAT 5A: Homework 03

Summer Session B 2025, with *Annie Adams*

1. Consider the random variable X with the following probability mass function (p.m.f):

k	- 2.4	-0.2	0	4
P(X = k)	0.1	0.4	a	0.2

where **a** is an as-of-yet unknown constant.

(a) What is the value of **a**?

$$0.1 + 0.4 + a + 0.2 = 1 \Rightarrow a = 1 - (0.1 + 0.4 + 0.2) = 0.3$$

(b) What is $\mathbb{P}(-2.4 \leq X < 0.2)$?

$$\mathbb{P}(-2.4 \leq X \leq 0.2) = \mathbb{P}(X = -2.4) + \mathbb{P}(X = -0.2) + \mathbb{P}(X = 0) = 0.1 + 0.4 + 0.3 = 0.8$$

(c) What is $\mathbb{P}(X \geq 0)$?

$$\mathbb{P}(X \geq 0) = \mathbb{P}(X = 0) + \mathbb{P}(X = 4) = 0.3 + 0.2 = 0.5$$

(d) What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \sum_{\text{all } k} k \cdot \mathbb{P}(X = k)$$

$$= (-2.4) \cdot \mathbb{P}(X = -2.4) + (0.2) \cdot \mathbb{P}(X = 0.2) + (0) \cdot \mathbb{P}(X = 0) + (4) \cdot \mathbb{P}(X = 4)$$

$$= (-2.4) \cdot (0.1) + (0.2) \cdot (0.4) + (0) \cdot (0.3) + (4) \cdot (0.2) = 0.48$$

(e) What is $\text{Var}(X)$?

If we use the second formula for variance, we first compute

$$\sum_{\text{all } k} k^2 \cdot \mathbb{P}(X = k) = (-2.4)^2 \cdot \mathbb{P}(X = -2.4) + (0.2)^2 \cdot \mathbb{P}(X = 0.2) + (0)^2 \cdot \mathbb{P}(X = 0) + (4)^2 \cdot \mathbb{P}(X = 4)$$

$$= (-2.4)^2 \cdot (0.1) + (0.2)^2 \cdot (0.4) + (0)^2 \cdot (0.3) + (4)^2 \cdot (0.2) = 3.792$$

$$\text{Var}(X) = \sum_{\text{all } k} k^2 \cdot \mathbb{P}(X = k) - (\mathbb{E}[X])^2 = 3.792 - (0.48)^2 = 3.5616$$

If we instead used the first formula for variance, we compute

$$\text{Var}(X) = \sum_{\text{all } k} (k - \mathbb{E}[X])^2 \cdot \mathbb{P}(X = k)$$

$$= (-2.4 - 0.48)^2 \cdot \mathbb{P}(X = -2.4) + (0.2 - 0.48)^2 \cdot \mathbb{P}(X = 0.2) + (0 - 0.48)^2 \cdot \mathbb{P}(X = 0) + (4 - 0.48)^2 \cdot \mathbb{P}(X = 4)$$

$$= (-2.4 - 0.48)^2 \cdot (0.1) + (0.2 - 0.48)^2 \cdot (0.4) + (0 - 0.48)^2 \cdot (0.3) + (4 - 0.48)^2 \cdot (0.2) = 3.5616$$

2. The weight of a randomly-selected fish from *Lake Gaucho* (in pounds) is normally distributed with mean 4 lbs and standard deviation 1.3 lbs. A fish is selected at random from *Lake Gaucho* and its weight is recorded.

(a) Define the random variable of interest.

Let X = the weight of a randomly-selected fish from *Lake Gaucho*.

(b) What is the probability that this fish weighs less than 2 pounds?

We know that $X \sim N(4, 1.3)$, and we seek $\mathbb{P}(X < 2)$; thus, we first standardize and then utilize our lookup table:

$$\mathbb{P}(X < 2) = \mathbb{P}\left(\frac{X - 4}{1.3} < \frac{2 - 4}{1.3} \right) = \mathbb{P}\left(\frac{X - 4}{1.3} < -1.53 \right) = 0.0618$$

(c) What is the probability that this fish weighs between 2.5 lbs and 3.8 lbs?

We seek $\mathbb{P}(2.5 \leq X \leq 3.8)$. Our strategy for computing quantities like this is always to convert everything to be in terms of left-tail areas, and then standardize:

$$\mathbb{P}(2.5 \leq X \leq 3.8) = \mathbb{P}(X \leq 3.8) - \mathbb{P}(X \leq 2.5)$$

$$= \mathbb{P}\left(\frac{X - 4}{1.3} < \frac{3.8 - 4}{1.3} \right) - \mathbb{P}\left(\frac{X - 4}{1.3} < \frac{2.5 - 4}{1.3} \right)$$

$$= \mathbb{P}\left(\frac{X - 4}{1.3} < -0.15 \right) - \mathbb{P}\left(\frac{X - 4}{1.3} < -1.15 \right)$$

(d) Suppose, now, that a random sample of 10 fish is caught (assume that the weights of fish in *Lake Gaucho* are independent), and the number of fish that weigh between 2.5 and 3.8 lbs is recorded. What is the probability that exactly 3 of these fish weigh between 2.5 and 3.8 lbs? **Hint:** you will need to define another random variable.

Following the hint, we let Y denote the number of fish, in a random sample of 10 fish caught from Lake Gaucho, that weigh between 2.5 and 3.8 lbs. Indeed, Y follows the Binomial distribution:

1) Independent Trials? Yes, since we are told to assume weights of different fish are independent.

2) Fixed Number of Trials? Yes; n = 10

3) Well-defined notion of 'success'? Yes; 'success' = 'weighing between 2.5 and 3.8 lbs'

4) Fixed probability of success? Yes; p = 0.3153, as found in part (d) above.

Therefore, we have $Y \sim \mathrm{Bin}(10, 0.3153)$ and so

$$\mathbb{P}(Y = 3) = \binom{10}{3}(0.3153)^3(1 - 0.3153)^{10-3} \approx 0.265 = 26.5\%$$

3. Suppose that 33% of a particular country's population has a college degree. A representative sample of 243 people is taken, and the proportion of these people who have a college degree is recorded.

(a) Define the parameter of interest.

Let *p* denote the proportion of the country's population that have a college degree.

(b) Define the random variable of interest. Use proper notation.

Let $\hat{P}$ denote the proportion of people in a representative sample of

100 that have a college degree.

(c) Check whether the success-failure conditions are satisfied.

In this case, we know the value of p: p = 0.33. Additionally, n = 243, so we check:

1. $np = (243)(0.33) = 80.19 \geq 10$

2. $n(1-p) = 162.81 \geq 10$

We see that both conditions are satisfied.

(d) What is the probability that over 30% of the sample have college degrees?

We seek $\mathbb{P}(\hat{P} \geq 0.3)$. By the Central Limit Theorem for Proportions (which we are able to invoke because the success-failure conditions are satisfied),

$$\hat{P} \sim N\left(0.33, \sqrt{\frac{(0.33)(1-0.33)}{243}}\right) \sim N(0.33, 0.0302)$$

Therefore, we compute

$$\mathbb{P}(\hat{P} > 0.3) = 1 - \mathbb{P}(\hat{P} \leq 0.3) = 1 - \mathbb{P}\left(\frac{\hat{P} - 0.33}{0.0302} \leq \frac{0.3 - 0.33}{0.0302}\right)$$

$$= 1 - \mathbb{P}(Z \leq -0.99)$$

where $Z \sim N(0,1)$. From a normal table, we therefore see that the desired probability is $1 - 0.1611 = 0.8389 = 83.89\%$

(e) What is the probability that the proportion of people in the sample with college degrees lies within 5% of the true proportion of 33%?

We now seek $\mathbb{P}(0.28 \leq \hat{P} \leq 0.38)$, which we compute as

$$\mathbb{P}(0.28 \leq \hat{P} \leq 0.38) = \mathbb{P}(\hat{P} \leq 0.38) - \mathbb{P}(\hat{P} \leq 0.28)$$

$$= \mathbb{P}\left(\frac{\hat{P} - 0.33}{0.0302} \leq \frac{0.38 - 0.33}{0.0302} \right) - \mathbb{P}\left(\frac{\hat{P} - 0.33}{0.0302} \leq \frac{0.28 - 0.33}{0.0302} \right)$$

$$= \mathbb{P}\left(\frac{\hat{P} - 0.33}{0.0302} \leq \frac{0.05}{0.0302} \right) - \mathbb{P}\left(\frac{\hat{P} - 0.33}{0.0302} \leq \frac{-0.05}{0.0302} \right)$$

$$= \mathbb{P}(Z \leq 1.66) - \mathbb{P}(Z \leq -1.66) = 0.9515 - 0.0485 = 0.903 = 90.3\%$$